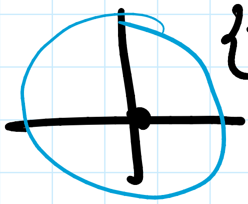
Algebraic links

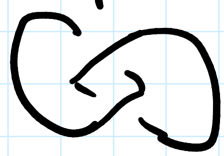
$C = \{f = 0\} \subset \mathbb{C}^2$

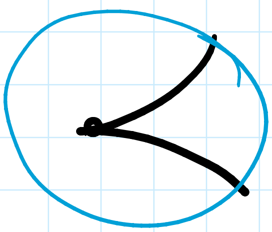
plane curve singularity (reduced for now)

$L = C \cap S^3_\varepsilon$ ← small sphere w. center at singularity

" algebraic link

Ex  $\{xy = 0\} \longleftrightarrow$ Hopf link



Ex  $\{x^2 = y^3\} \longleftrightarrow$ trefoil $T(2,3)$

$\{x^m = y^n\} \longleftrightarrow T(m,n)$ torus link.

Fact • Irreducible components of C

$\updownarrow$

connected components of the link L

• C reduced, irreducible $\Rightarrow$ L is a knot
with 1 component

In this case, L is an iterated

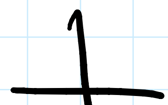
In this case, L is an iterated
cable of a torus knot, full classification related to Puiseaux expansion of C

- For general links more complicated, see the book by Eisenbud-Neumann.

<u>Conj</u> (Oblomkov, Rasmussen, Shende)

$$HHH^0(L) = \widehat{\bigoplus_{k=0}} H^*(\mathrm{Hilb}^k(C,0))$$

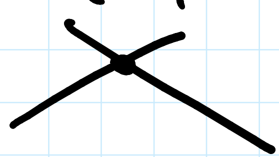
bigraded by k and homological degree

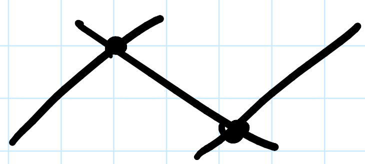
<u>Ex</u> $C = \{xy=0\}$ 

$\mathrm{Hilb}^0(C,0) = pt = \{\mathcal{O}\}$

$\mathrm{Hilb}^1(C,0) = pt = \{m\}$

$\mathrm{Hilb}^2(C,0) = \mathrm{Hilb}^2(\mathbb{C}^2,0) = \mathbb{P}^1$

$\mathrm{Hilb}^3(C,0) =$  two lines glued at a pt

$\mathrm{Hilb}^4(C,0) =$ 

$\vdots$

$\mathrm{Hilb}^k(C,0) =$ chain of $(k-1)$ lines.

$$H^0 = \mathbb{C} \quad H^2 = \mathbb{C}^{k-1}$$

$$1 + q + q^2(1+t^2) + q^3(1+2t^2) + \dots$$

$$1+q+q^2(1+t^2)+q^3(1+2t^2)+\dots$$

a Poincaré series

$$=\frac{1}{1-q}+\frac{q^2t^2}{(1-q)^2}$$

<u>Compare:</u> $R \oplus \frac{R}{(x_1-x_2)}$

$R=\mathbb{C}[x_1,x_2]$.

Wide open in general!!

What is known:

- $\sum q^k \chi(\mathrm{Hilb}^k(C,0)) = \mathrm{HOMFLY\text{-}PT}(q, a=0)$ (Maulik)

- For torus knots, can compute both sides

HHH computed by recursions from lecture 1 (Hogancamp-Mellit)

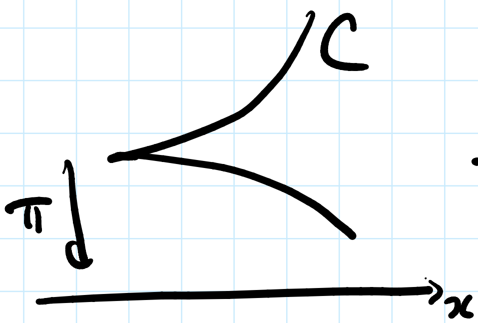
$\longleftrightarrow$

$\mathrm{Hilb}^k(C,0)$ has a paving by affine cells, combinatorial formula for dim (ORS + more later)

- Recall that for an r-component link HHH has an action of $\mathbb{C}[x_1 \dots x_r]$
  - For $r=1$, constructed by Maulik-Yun, Migliorini-Shende, Rennemo
  - For $r>1$, constructed by Kivinen

- For $r>1$, constructed by Kivinen
- Roughly speaking, x_i adds a point on i-th component of C.

(need versal deformation to make it work)



Choose a projection of C to some line, degree $= n$

$C =$ "spectral curve"

$$\pi_* \mathcal{O}_{C,0} = \text{rank } n \text{ free } \mathbb{C}[[x]]\text{-module}$$
$$\simeq \mathbb{C}^n[[x]] \circlearrowleft Y$$

$Y =$ multiplication by y

<u>Ex</u> $C = \{x^2 = y^3\}$

$$\mathcal{O}_{C,0} = \mathbb{C}[[x]]\langle 1, y, y^2\rangle \quad Y = \begin{pmatrix} 0 & 0 & x^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{O}_{C,0} = \mathbb{C}[[y]]\langle 1, x\rangle$$
$$X = \begin{pmatrix} 0 & y^3 \\ 1 & 0 \end{pmatrix}$$

different descriptions of the same curve.

Equation of $C \longleftrightarrow$ characteristic polynomial

Affine <u>Springer fiber</u> of C

Affine Springer fiber of C

$$Sp_Y = \left\{ \underset{\text{lattice}}{V \subset \mathbb{C}^n((x))} \;\middle|\; \begin{matrix} xV \subset V \\ YV \subset V \end{matrix} \right\} \subset Gr$$

affine Grassmannian

Facts: • If C is irreducible,

$Sp_Y \triangleq$ compactified Jacobian of C

$= \{$ rank 1 torsion free sheaves on $C \}$

$\simeq \mathrm{Hilb}^N(C,0)$ for $N \gg 0$.

does not depend on projection

• Thm (Migliorini–Shende, Maulik–Yun)

$$\bigoplus_{k\geq 0} H^\bullet(\mathrm{Hilb}^k(C,0)) = gr_P^\bullet H^\bullet(Sp_Y) \otimes \mathbb{C}[x]$$

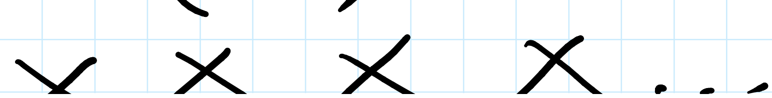
"perverse filtration"

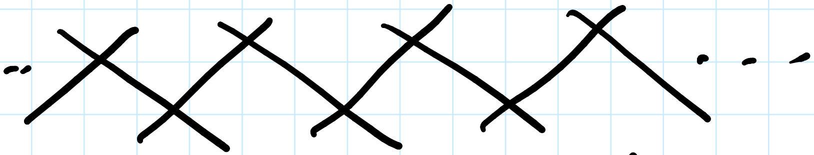
+ there's an action of $\mathfrak{sl}_2$ on RHS satisfying "curious hard lefschetz" with respect to perverse filtration

(compare with "curious hard lefschetz" for weight filtration from Lecture 3

Ex $\{x^2 = y^2\}$ $\quad Y = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix}$

$Sp_Y =$



$Sp_\gamma =$ ...  ...

Ex $\{x^2 = y^3\}$ $\quad Sp_\gamma = \mathbb{P}^1 =$ compactified Jacobian

Ex $\{x^m = y^n\}$, $GCD(m,n) = 1$

Paved by affine cells, explicit combinatorial parametrization (q,t - Catalan)
(Lusztig - Smelt, Piontkowski, G.-Mazin-Vazirani, Hikita..)

$\{x^3 = y^4\} \longrightarrow$ cone over Hirzebruch surface

H^0	H^2	H^4	H^6
1	1	1	1
		1	

$\updownarrow$ perverse filtration

Compare w. E_6 picture from Lecture 3

Conj (Shende) $H^*(\text{braid variety}) \simeq H^*(Sp_\gamma)$
+ weight filtration $\qquad$ + perverse filtration

Thm (Oblomkov-Yun)

For $\{x^m = y^n\}$ $\quad gcd(m,n) = 1$

For $\{x^m = y^n\}$, $\gcd(m,n)=1$

$H^*(Sp_\gamma)$ is generated by tautological classes + explicit relations.

<u>Thm</u> (Kivinen) $\{x^{kn} = y^n\}$

$$Y \sim \begin{pmatrix} \zeta_1 x^k & & 0 \\ & \ddots & \\ 0 & & \zeta_n x^k \end{pmatrix} \qquad \zeta_i = \text{root of unity}$$

- $\mathbb{Z}^{n-1}$ acts on Sp_Y by translations
- $(\mathbb{C}^*)^n$ stabilizes Y and acts on Sp_Y

$\Rightarrow$ $H^*(Sp_Y)$ matches $HHH^0(T(n,kn))$

$\mathbb{Z}^{n-1} \longleftrightarrow x_1 \dots x_n$

$H^*_{(\mathbb{C}^*)^n}(Sp_Y)$ matches $HY^0(T(n,kn))$

$y_1 \dots y_n$ = equivariant parameters.

deformed homology from last lecture.

<u>Thm</u> (Garner-Kivinen)

(a) For <u>any</u> curve C (not necessary reduced!)

$\bigsqcup_k \mathrm{Hilb}^k(C,0)$ = generalized affine Springer fiber for $G = GL_n$, $N = \mathfrak{gl}_n \oplus \mathbb{C}^n$

Springer fiber for $G = GL_n, N = \mathfrak{gl}_n \oplus \mathbb{C}^n$

(b) There is an action of BFN Coulomb branch algebra for (G,N) in $\bigoplus_k H^*(\mathrm{Hilb}^k(C,0))$

(c) If C admits a $\mathbb{C}^*$ action (say, $\{x^m = y^n\}$) then there is an action of quantized BFN algebra for (G,N) in $\bigoplus_k H^*_{\mathbb{C}^*}(\mathrm{Hilb}^k(C,0))$

Rmk (Kodera-Nakajima)

quantized BFN algebra for $(G,N) = (GL_n, \mathfrak{gl}_n \oplus \mathbb{C}^n)$ = rational Cherednik algebra.